

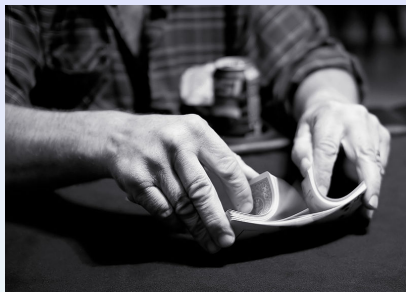
# Card Shuffles & Cantor space: an inverse semigroup perspective

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# The intuition / motivation

Card shuffles are well-studied in *combinatorics, probability, representation theory, statistics, &c.*



*Credit: Johnny Blood Photography*

There are also applications in theoretical computer science.  
These are best understood via (inverse) semigroup theory.

# Why Computer Science?

## One example: Race Conditions

*“A parent process spawns several child processes, each of which competes for the parent’s resources. These requests must be dealt with in order, one at a time. The outcome varies depending on the order in which these are processed”*

This is the motivation. However, today’s talk is about the **semigroup theory**. Any applications are side-effects!

# How to shuffle two (possibly infinite) decks of cards

## The Riffle Shuffle

- Cards from Deck  $A$  and Deck  $B$  are merged into a single stack.
- At each step, a single card is taken from the bottom of either  $A$  or  $B$ , and placed on top of the stack.

### Some important conventions:

- The ordering of cards is preserved.
- Every card from each deck ends up in the stack.

# Modeling Riffle Shuffles

We model a **deck** of  $a$  cards by the well-ordered set

$$[0, a) = \{n \in \mathbb{N} : n < a\}$$

(We allow for  $a = \infty$  in this definition, giving  $[0, \infty) = \mathbb{N}$ ).

A **pair of decks** is modeled by the disjoint union

$$[0, a) \uplus [0, b) = [0, a) \times \{0\} \cup [0, b) \times \{1\}$$

equipped with the induced partial order

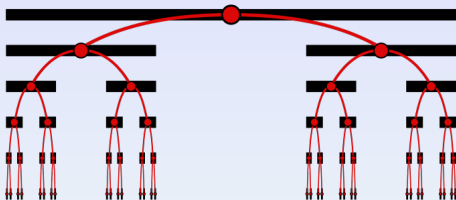
$$(x, i) \leq (y, j) \text{ iff } i = j \text{ and } x \leq y$$

A **riffle shuffle** is then an order-preserving bijection:

$$\phi : [0, a) \uplus [0, b) \rightarrow [0, a + b)$$

# The infinitary setting

Every riffle shuffle of two infinite decks uniquely determines & is determined by a point of the Cantor set  $\mathcal{C}$ .



Formally, one-sided countably infinite strings over  $\{0, 1\}$ , or equivalently,  $\mathcal{C} = \mathbf{Set}(\mathbb{N}, \{0, 1\})$ .

## Computational motivation

*Infinite shuffles model potentially non-terminating processes.*

# The correspondence:

Given a shuffle of two infinite decks

$$\phi : [0, \infty) \uplus [0, \infty) \rightarrow [0, \infty)$$

we define the corresponding Cantor point  $p_\phi \in \mathcal{C}$  by

$$p_\phi(n) = \begin{cases} 0 & n = \phi(x, 0) \text{ for some } x \in \mathbb{N} \\ 1 & n = \phi(x, 1) \text{ for some } x \in \mathbb{N} \end{cases}$$

Operationally:  $p_\phi$  is an **instruction**

At the  $n^{\text{th}}$  step, take the next card from:

- The first deck, when  $p_\phi(n) = 0$
- The second deck, when  $p_\phi(n) = 1$

# An illustrative example

The **perfect riffle shuffle**:

Cards are alternately taken from each deck

corresponds to the **alternating Cantor point**  $a(n) = n \pmod{2}$ .

$$a = 0101010101\dots$$

Not all Cantor points determine shuffles:

We require a Cantor point  $c \in \mathcal{C}$  to satisfy:

$$\sum_{j=0}^{\infty} c(j) = \infty = \sum_{j=0}^{\infty} 1 - c(j)$$

For the condition, “every card is played at some point”.



# An inverse semigroup approach ...

We will be mixing order theory & partiality.

By analogy with Kleene equality

In a poset we write  $f(a) \lesssim g(b)$  for

“ $f(a) \leq g(b)$  provided both  $f(a)$  and  $g(b)$  are defined”.

A partial injection  $f : (P, \leq) \rightarrow (Q, \leq)$  is

- **monotone (mono.)** when  $a \leq b \Rightarrow f(a) \lesssim f(b)$ ,
- **anti-monotone (anti)** when  $a \leq b \Rightarrow f(b) \lesssim f(a)$ .

# Monos, antis. and composition

## Notation:

Denote the monotone partial injections from  $P$  to  $Q$  by  $\mathbf{mono}(P, Q)$ , and the anti-monotone partial injections from  $P$  to  $Q$  by  $\mathbf{anti}(P, Q)$ .

Given partial injections:

- $m \in \mathbf{mono}(P, Q)$  and  $n \in \mathbf{mono}(Q, R)$
- $a \in \mathbf{anti}(P, Q)$  and  $b \in \mathbf{anti}(Q, R)$

then  $ba, nm \in \mathbf{mono}(P, R)$  and  $na, bm \in \mathbf{anti}(P, R)$ .

## Composing monotone & anti-monotone partial injections

|       |       |       |
|-------|-------|-------|
|       | mono. | anti. |
| mono. | mono. | anti. |
| anti. | anti. | mono. |

# A word of warning!

Monotone partial injections form categories / monoids  
— *these are not generally inverse categories / monoids.*

## A simple counterexample:

Consider the successor function

$$\text{succ} \in \mathbf{mono}((\mathbb{N}, =), (\mathbb{N}, \leq))$$

This is monotone, but its generalised inverse certainly is not!

# A simple and relevant setting:

Denote the set of monotone partial injections on  $\mathbb{N}$  by

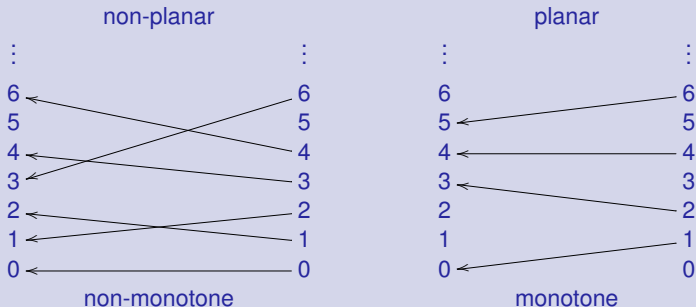
$$\mathbf{mono}(\mathbb{N}, \mathbb{N}) \leq \mathbf{plnj}(\mathbb{N}, \mathbb{N})$$

This set is closed under composition and generalised inverse and so forms an inverse monoid.

# Some intuition ...

Think of monotone partial injections on  $\mathbb{N}$  as 'planar diagrams'.

## Monotonicity as planarity for partial injections on $\mathbb{N}$



Planarity is a **big deal** in many areas of C.S.

# Why planarity?

- 1 The **quantum Jones polynomial** algorithm (Aharonov, Jones, Landau)
  - A QM algorithm for computing Jones polynomials at  $e^{\frac{2k\pi i}{5}}$
  - Classically, a (presumably)  $P\#$  problem.
  - Based on the *Temperley-Lieb algebra*  
“Knot theory without crossings” – L. Kaufmann.
- 2 **Lambek pregroups** (From categorical linguistics)
  - Becoming used Natural Language Processing
  - Diagrams determined by *planarity & acyclicity*.
- 3 **Complexity theory** (Planarity provides bounds to complexity).
  - *Matchgates and classical simulation of quantum circuits*  
– R. Jozsa, A. Miyake
  - Restricting swap gates allows for *efficient classical simulation* of QM circuits.
- 4 ...

First ... some simple theory!



# Characterising monotone partial injections on $\mathbb{N}$

Graphically, or otherwise, the following is straightforward:

Every  $f \in \text{mono}(\mathbb{N}, \mathbb{N})$  is uniquely determined by its initial & final idempotents.

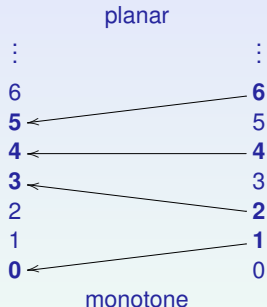


$f^{-1}f$  and  $ff^{-1}$  are partial identities on  $\mathbb{N}$ .

# Characterising monotone partial injections on $\mathbb{N}$

Graphically, or otherwise, the following is straightforward:

Every  $f \in \text{mono}(\mathbb{N}, \mathbb{N})$  is uniquely determined by its initial & final idempotents on the well-ordered set  $\mathbb{N}$ .



$f^{-1}f$  and  $ff^{-1}$  are partial identities on  $\mathbb{N}$ .

# Idempotents as Cantor points

Indicator functions for subsets of  $\mathbb{N}$  are points of the Cantor set.

Abusing notation slightly: given  $e^2 = e = 1_E \in \mathbf{mono}(\mathbb{N}, \mathbb{N})$ , we write  $Ind_e : \mathbb{N} \rightarrow \{0, 1\}$ , or  $Ind_e \in \mathcal{C}$ .

A trivial observation:

For a monotone partial injection  $f \in \mathbf{mono}(\mathbb{N}, \mathbb{N})$ ,

$$\sum_{n=0}^{\infty} ind_{ff^{-1}}(n) = \sum_{n=0}^{\infty} ind_{f^{-1}f}(n) \in \mathbb{N} \cup \{\infty\}$$

# A few simple definitions

A pair of Cantor points  $(d, c) \in \mathcal{C} \times \mathcal{C}$  is **balanced** when

$$\sum_{j=0}^{\infty} d(j) = \sum_{j=0}^{\infty} c(j) \in \mathbb{N} \cup \{\infty\}$$

We denote the set of balanced Cantor pairs by  $\mathfrak{B} \subseteq \mathcal{C} \times \mathcal{C}$ .

There is a 1 : 1 correspondence  $\mathfrak{B} \equiv \mathbf{mono}(\mathbb{N}, \mathbb{N})$ .

# Giving this explicitly:

Balanced Cantor pairs  $\equiv$  monotone partial injections

- $\Leftarrow$  Given  $f \in \mathbf{mono}(\mathbb{N}, \mathbb{N})$ , the balanced pair is:

$$(Ind_{ff^{-1}}, Ind_{f^{-1}f}) \in \mathfrak{B}$$

- $\Rightarrow$  Given  $(t, s) \in \mathfrak{B}$ , define  $m_{(t,s)} \in \mathbf{mono}(\mathbb{N}, \mathbb{N})$  by

$$m_{(t,s)}(n) = \begin{cases} \perp & s(n) = 0 \\ \min_{x \in \mathbb{N}} \left\{ \sum_{j=0}^x t(j) = \sum_{j=0}^n s(j) \right\} & s(n) = 1 \end{cases}$$

# An illustration:

A balanced pair of Cantor points:

$$t = 1001110 \text{ , } s = 0110101 \dots$$

| $n =$    | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|----------|---|---|---|---|---|---|---|-----|
| $s(n) =$ | 0 | 1 | 1 | 0 | 1 | 0 | 1 | ... |
| $t(n) =$ | 1 | 0 | 0 | 1 | 1 | 1 | 0 | ... |

# An illustration:

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$$t = 1001110 \text{ , } s = 0110101 \dots$$

| $n =$                    | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|--------------------------|---|---|---|---|---|---|---|-----|
| $s(n) =$                 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | ... |
| $\sum_{j \leq n} s(j) =$ | 0 | 1 | 2 | 2 | 3 | 3 | 4 | ... |
| $\sum_{j \leq n} t(j) =$ | 1 | 1 | 1 | 2 | 3 | 4 | 4 | ... |
| $t(n) =$                 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | ... |

# A monoid operation on balanced pairs?

There exists some composition operation  $\cdot : \mathfrak{B} \times \mathfrak{B} \rightarrow \mathfrak{B}$  such that  $(\mathfrak{B}, \cdot) \cong \mathbf{mono}(\mathbb{N}, \mathbb{N})$ .

What does this look like?



# Normal forms (I)

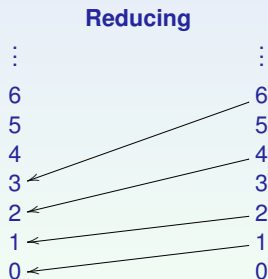
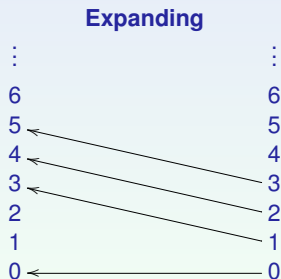
A monotone partial injection is **reducing** when

$$ff^{-1} = 1_{[0,x)} \quad \text{for some } x \in \mathbb{N} \cup \{\infty\}$$

Dually, it is **expanding** when

$$ff^{-1}f = 1_{[0,x)} \quad \text{for some } x \in \mathbb{N} \cup \{\infty\}$$

An illustrative example:



# Cantor points as reducing / expanding arrows

Reducing (resp. expanding) arrows are uniquely determined by their initial (resp. final) idempotents.

Given  $c \in \mathcal{C}$ , define  $Red_c \in Mono(\mathbb{N}, \mathbb{N})$  by

$$Red_c(n) = \begin{cases} \perp & n = 0 \\ \sum_{j=0}^n c(n) - 1 & n = 1 \end{cases}$$

Dually, define  $Exp_c \in Mono(\mathbb{N}, \mathbb{N})$  by

$$Exp_c = Red_c^{-1}$$

# Normal forms (II)

Given arbitrary  $f \neq 0 \in \mathbf{mono}(\mathbb{N}, \mathbb{N})$ ,  
then the balanced pair  $(t, s) = (Ind_{ff^{-1}}, Ind_{f^{-1}f}) \in \mathfrak{B}$   
is the unique balanced pair satisfying

$$f = Exp_t Red_s$$

(The only non-trivial point is uniqueness, which follows  
since  $(t, s)$  is required to be balanced).

# By considering normal forms (or directly)

Given  $(v, u), (t, s) \in \mathfrak{B}$ , define a composition by:

$$(v, u) \cdot (t, s) = \begin{cases} (\mathbf{0}, \mathbf{0}) & t(n)u(n) = 0 \quad \forall n \in \mathbb{N} \\ (x, w) & \text{otherwise} \end{cases}$$

where  $w(n) = s(n).u(j).t(j) \in \{0, 1\}$ ,

$$j = \min_{j \in \mathbb{N}} \left\{ \sum_{\alpha=0}^j t(\alpha) = \sum_{\alpha=0}^n s(\alpha) \right\}$$

and similarly,  $x(n) = v(n).u(k).t(k) \in \{0, 1\}$ ,

$$k = \min_{k \in \mathbb{N}} \left\{ \sum_{\alpha=0}^k u(\alpha) = \sum_{\alpha=0}^n v(\alpha) \right\}$$

The generalised inverse is immediate:  $(t, s)^{-1} = (s.t)$ .

This gives  $(\mathfrak{B}, \cdot) \cong \mathbf{mono}(\mathbb{N}, \mathbb{N})$  as required.

# Duals and self-encodings

Recall the complement / dual operation on the Cantor set:

$$c^\perp(n) = c(n) + 1 \pmod{2} \quad \forall c \in \mathcal{C}$$

(e.g.  $c = 0100101 \dots$  has complement  $c^\perp = 1011010 \dots$ ).

## A key definition

An element  $(b, a) \in \mathfrak{B}$  is **complemented** when  $(b^\perp, a^\perp) \in \mathfrak{B}$ ,  
and is **dual-inverse** when  $(b, a)^{-1} = (b^\perp, a^\perp)$ .

# Duals, inverses, and shuffles

Fairly simply,  $(b, a) \in \mathfrak{B}$  is dual-inverse iff

$$b = a^\perp \text{ and } \sum_{n=0}^{\infty} a(n) = \infty = \sum_{n=0}^{\infty} 1 - a(n)$$

There is then a bijective correspondence between dual-inverse arrows of  $\mathfrak{B}$ , and riffle shuffles of two infinite decks of cards.

These are both determined by Cantor points  $a \in \mathcal{C}$  satisfying

$$\sum_{n=0}^{\infty} a(n) = \infty = \sum_{n=0}^{\infty} 1 - a(n)$$

We call these **dual-balanced Cantor points**.

# From D.-B. Cantor points to Young Tableaux

There is an correspondence between

- 1 D.-B. Cantor points,
- 2 Shuffles of infinite decks of cards,
- 3  $(\infty, \infty)$  Young tableaux.

$$c = 1010101100 \dots \in \mathcal{C}$$

|            |   |   |   |   |   |     |
|------------|---|---|---|---|---|-----|
| $c(n) = 0$ | 1 | 3 | 5 | 8 | 9 | ... |
| $c(n) = 1$ | 0 | 2 | 4 | 6 | 7 | ... |

The obvious question:

What about **standard** Young tableaux?

*This is where we start to need the inverse semigroup theory.*

# D.-B. Cantor points as inverse monoids

**Proposition:** There is a 1:1 correspondence between:

- Dual-balanced Cantor points,
- Effective representations of the 2-generator polycyclic monoid within  $\mathbf{mono}(\mathbb{N}, \mathbb{N})$ .

Recall – the polycyclic monoid  $P_2$

- Two generators,  $\{p, q\}$
- Relations:

$$pq^{-1} = 0 = q^{-1}p \text{ and } pp^{-1} = 1 = qq^{-1}$$

Useful fact: polycyclic monoids are *congruence-free*.



# Monotone representations of $P_2$

Let  $c \in \mathcal{C}$  be dual-balanced. Looking at normal forms,

$$\text{Exp}_c \text{Red}_{c^\perp} = (c, c^\perp) \quad \text{and} \quad (c^\perp, c) = \text{Exp}_{c^\perp} \text{Red}_c$$

By construction,  $\text{Red}_c \text{Exp}_c = 1_{\mathbb{N}} = \text{Red}_{c^\perp} \text{Exp}_{c^\perp}$ .

By definition, of  $(\ )^\perp : \mathcal{C} \rightarrow \mathcal{C}$ ,

$$c(n) = 0 \iff c^\perp(n) = 1$$

and so

$$\text{Red}_c \text{Exp}_{c^\perp} = 0_{\mathbb{N}} = \text{Red}_{c^\perp} \text{Exp}_c$$

The assignment  $p \mapsto \text{Red}_c$ ,  $q \mapsto \text{Red}_{c^\perp}$  gives an effective monotone representation of  $P_2$ .

— all effective monotone representations arise in this way.

# As always ... an example

For the *alternating Cantor point*, or *perfect riffle shuffle*,

$$a(n) = n \pmod{2} \quad \text{or } a = 0101010101\dots$$

we derive the representation of  $P_2$  corresponding to the Cantor pairing:

$$p^{-1}(x) = 2x \quad \text{and} \quad q^{-1}(x) = 2x + 1$$

# On to standard Young tableaux

In **standard** Young tableaux, the cells are well-ordered both *horizontally* and *vertically*.

|     |     |
|-----|-----|
| $x$ | $a$ |
| $y$ | $b$ |

$$\begin{array}{ccc} x & \leq & a \\ \downarrow & & \downarrow \\ b & \leq & y \end{array}$$

Horizontal ordering corresponds to monotonicity.

What about the vertical ordering?

# Some standard(?) semigroup theory

A (binary) **ballot sequence** is an element  $w \in \{0, 1\}^*$  where, for every prefix  $u$  of  $w$ ,

$$\#1s \text{ in } u \leq \#0s \text{ in } u$$

Denote the set of all finite ballot sequences by *Ballot* — this forms a submonoid of  $\{0, 1\}^*$ .

**By contradiction:** Consider  $v, w \in \textit{Ballot}$  such that  $vw \notin \textit{Ballot}$ . Then there exists some prefix  $u$  of  $vw$  satisfying  $\#0s \text{ in } u < \#1s \text{ in } u$ . As  $v \in \textit{Ballot}$ ,  $u$  is not a prefix of  $v$ , so  $u = vl$ , for some prefix  $l$  of  $w$ . However,  $\#0s \text{ in } v \geq \#1s \text{ in } v$ . Therefore,  $\#1s \text{ in } l \geq \#0s \text{ in } l$ , contradicting the assumption that  $w \in \textit{Ballot}$ .

# A deceptively simple monoid

Ballot sequences are *well-studied* in combinatorics – but also make for interesting monoids!

**Proposition** The monoid of binary ballot sequences is not finitely generated.

**By contradiction:** Assume a finite generating set  $G$  for  $\text{Ballot} \leq \{0, 1\}^*$ . As  $G$  is finite, the longest contiguous string of  $1$ s in any member of  $G$  is bounded by some finite  $K \in \mathbb{N}$ . No composite of members of  $G$  can account for the ballot sequence  $0^{K+1}1^{K+1}$ .

# From the finite to the infinite:

A Cantor point  $c \in \mathcal{C}$  is **ballot** when every prefix is a member of the Ballot monoid.

$$\sum_{j=0}^N c(j) \leq \sum_{j=0}^N c^\perp(j) \quad \forall N \in \mathbb{N}$$

Denote the ballot Cantor points by  $\mathcal{S} \subseteq \mathcal{C}$ .

## Question:

How do such Cantor points behave under the point-wise partial order:

$$a \leq b \text{ iff } a(n) \leq b(n) \quad \forall n \in \mathbb{N}$$

# The ballot Scott domain

## Key properties:

- There is no top element & they are **not** closed under joins  
 $(c \vee d)(n) = \max\{c(n), d(n)\}$ .
- They **are** closed under the meet,  $(c \wedge d)(n) = c(n)d(n)$
- There is a bottom element  $\perp(n) = 0$ , for all  $n \in \mathbb{N}$ .
- The supremum of every chain  $c_0 \leq c_1 \leq c_2 \leq \dots$  is also in  $\mathcal{B}$ 
  - chain-completeness  $\Rightarrow$  directed completeness, assuming the axiom of choice (Iwamura's Lemma).
- There is a notion of **finite support** / **compactness**:  $c \in \mathcal{B}$  is “**compact**” iff  $\sum_{j=0}^{\infty} c(j) < \infty$ , and every element is the supremum of a chain of such elements.

## Scott Domains ...

- Introduced by Dana Scott (early 1970s) to model pure untyped  $\lambda$  calculus
  - and hence **computational universality**.
- Also used for semantics of **functional programming** languages, due to the existence of solutions of arbitrary **fixed-point equations**.

This particular Scott domain is  
a subset of Cantor space.  
We can draw a picture.





# Combining two properties:

A **dual-balanced ballot** Cantor point  $c \in \mathcal{C}$  satisfies:

- $\sum_{j=0}^{\infty} c(j) = \sum_{j=0}^{\infty} c^{\perp}(j)$
- $\sum_{j=0}^N c(j) \leq \sum_{j=0}^N c^{\perp}(j)$ .

There is a 1:1 correspondence:

DBB Cantor points  $\equiv$  Standard  $(\infty, \infty)$  Young tableaux

These are given by:

Removing the 'compact' points from the ballot Scott domain.

# The motivation

As card-shuffling:

The only way we can see:  $\dots$ 

|     |     |
|-----|-----|
|     | $x$ |
| $y$ | $z$ |

 with  $z \leq x$  is when

*“More cards have been laid from Deck  $B$  than from Deck  $A$ ”*

As DBB Cantor points are dual-balanced, they uniquely determine representations of  $P_2$ , as monotone partial injections on  $\mathbb{N}$ .

Call these **standard monotone representations**.

# Some computer science motivation

Recall the motivation for studying Shuffles, from *race conditions*.

- Operations from Process A push data onto a stack.
- Operations from Process B pop data off a stack.
- The Ballot condition prevents us from trying to *read data from an empty stack*.

# Fun & games with polycyclic monoids

A very standard result **N. & P. (1970)**

There exists an embedding of  $P_\infty$  into  $P_2$ .

Recall:

The infinite-generator polycyclic monoid  $P_\infty$  has generating

set  $\{p_j\}_{j \in \mathbb{N}}$ , with relations  $p_j p_k^{-1} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$

The embedding is given by

$$p_j \mapsto pq^j \quad , \quad p_j^{-1} \mapsto q^{-j}p^{-1}$$

Straightforward to check that the required relations are satisfied!

# Polycyclic monoids as bijections

A slightly lesser-known result **PH & MVL** ( ... a while back)

Representations of  $P_\infty$  within  $\mathbf{plnj}(\mathbb{N}, \mathbb{N})$  correspond to injections

$$\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$$

which are bijections when the representation is effective

## A very simple construction

For a given representation, we define

$$\psi(x, y) = p_x^{-1}(y) \quad \forall (x, y) \in \mathbb{N} \times \mathbb{N}$$

## A worked example:

Let's do this for the *standard monotone representation* determined by the *alternating Cantor point*  $a \in \mathcal{C}$ .

$$p^{-1}(n) = 2n \quad \text{and} \quad q^{-1}(n) = 2n + 1$$

Expanding out, we get

$$\Psi_a(x, y) = q^{-x} p^{-1}(y) = 2^{x+1}y + 2^x - 1$$

A (Hilbert-hotel style) bijection from  $\mathbb{N} \times \mathbb{N}$  to  $\mathbb{N}$ .

# The infinite perfect shuffle

View this as 'shuffling countably infinitely many decks of cards'.

| $x$      | $y =$ | 0        | 1        | 2        | 3        | 4        | 5        | ...      |
|----------|-------|----------|----------|----------|----------|----------|----------|----------|
| 0        |       | 0        | 2        | 4        | 6        | 8        | 10       | ...      |
| 1        |       | 1        | 5        | 9        | 13       | 17       | 21       | ...      |
| 2        |       | 3        | 11       | 19       | 27       | 35       | 43       | ...      |
| 3        |       | 7        | 23       | 39       | 55       | 71       | 87       | ...      |
| 4        |       | 15       | 47       | 79       | 111      | 143      | 175      | ...      |
| 5        |       | 31       | 94       | 159      | 223      | 287      | 351      | ...      |
| $\vdots$ |       | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ |



# A few observations:

## Simple observations

- This table contains every natural number.
- Both rows and columns appear to be well-ordered
  - an  $(\infty, \infty, \infty, \dots)$  standard Young tableau?
- There seems to be some ‘underlying fractal structure’ ...

More practically — how easy is it to perform this shuffle?

# Deep fractal structure ??

On the  $n^{\text{th}}$  step, we play from Deck  $x$ :

|          | $\dots$ | $Deck_4$ | $Deck_3$ | $Deck_2$ | $Deck_1$ | $Deck_0$ |
|----------|---------|----------|----------|----------|----------|----------|
| $n = 1$  |         |          |          |          |          | •        |
| $n = 2$  |         |          |          |          | •        |          |
| $n = 3$  |         |          |          |          |          | •        |
| $n = 4$  |         |          |          | •        |          |          |
| $n = 5$  |         |          |          |          |          | •        |
| $n = 6$  |         |          |          |          | •        |          |
| $n = 7$  |         |          |          |          |          | •        |
| $n = 8$  |         |          | •        |          |          |          |
| $n = 9$  |         |          |          |          |          | •        |
| $n = 10$ |         |          |          |          | •        |          |
| $n = 11$ |         |          |          |          |          | •        |
| $n = 12$ |         |          |          | •        |          |          |
| $n = 13$ |         |          |          |          |          | •        |
| $n = 14$ |         |          |          |          | •        |          |
| $n = 15$ |         |          |          |          |          | •        |
| $n = 16$ |         | •        |          |          |          |          |

# This looks kind of familiar!

|          | ... | $2^4$ | $2^3$ | $2^2$ | $2^1$ | $2^0$ |
|----------|-----|-------|-------|-------|-------|-------|
| $n = 1$  |     |       |       |       |       | 1     |
| $n = 2$  |     |       |       |       | 1     | 0     |
| $n = 3$  |     |       |       |       | 1     | 1     |
| $n = 4$  |     |       |       | 1     | 0     | 0     |
| $n = 5$  |     |       |       | 1     | 0     | 1     |
| $n = 6$  |     |       |       | 1     | 1     | 0     |
| $n = 7$  |     |       |       | 1     | 1     | 1     |
| $n = 8$  |     |       | 1     | 0     | 0     | 0     |
| $n = 9$  |     |       | 1     | 0     | 0     | 1     |
| $n = 10$ |     |       | 1     | 0     | 1     | 0     |
| $n = 11$ |     |       | 1     | 0     | 1     | 1     |
| $n = 12$ |     |       | 1     | 1     | 0     | 0     |
| $n = 13$ |     |       | 1     | 1     | 0     | 1     |
| $n = 14$ |     |       | 1     | 1     | 1     | 0     |
| $n = 15$ |     |       | 1     | 1     | 1     | 1     |
| $n = 16$ |     | 1     | 0     | 0     | 0     | 0     |

# Performing the perfect infinite riffle

## A very simple rule

- 1 Count in binary ...
- 2 Which bit has changed from 0 to 1?
- 3 Play a card from that deck!

# The standard Young property

It is straightforward that **rows** and **columns** are well-ordered:

|     |     |
|-----|-----|
| $k$ | $m$ |
| $l$ |     |

$k = \Psi_a(x, y)$  for some  
 $(x, y) \in \mathbb{N} \times \mathbb{N}$ .

- $l = 2k + 1 > k$
- $m = k + 2^{y+1} > k$ .

They also contain all natural numbers.

**Claim** These properties follow generally from:

- 1 The fact that representations of  $P_2$  are *monotone* (since they are derived from DB Cantor points).
- 2 The ballot property on these Cantor points.

# A quick outline

Let  $c \in \mathcal{C}$  be a dual-balanced ballot Cantor point. This determines an effective monotone representation  $P_2 \xrightarrow{\mathcal{C}} \mathbf{plnj}(\mathbb{N}, \mathbb{N})$  which corresponds to an  $(\infty, \infty)$  Young tableau:

|             |             |             |             |             |     |
|-------------|-------------|-------------|-------------|-------------|-----|
| $p^{-1}(0)$ | $p^{-1}(1)$ | $p^{-1}(2)$ | $p^{-1}(3)$ | $p^{-1}(4)$ | ... |
| $q^{-1}(0)$ | $q^{-1}(1)$ | $q^{-1}(2)$ | $q^{-1}(3)$ | $q^{-1}(4)$ | ... |

By the ballot property,  $p^{-1}(n) \leq q^{-1}(n)$ , so this is *standard*.

# A quick outline (cont.)

By the same properties,  $q^{-k}(n) < q^{-(k+1)}(n)$ , so the following table is has well-ordered rows and columns:

|                   |                   |                   |                   |                   |          |
|-------------------|-------------------|-------------------|-------------------|-------------------|----------|
| $p^{-1}(0)$       | $p^{-1}(1)$       | $p^{-1}(2)$       | $p^{-1}(3)$       | $p^{-1}(4)$       | ...      |
| $q^{-1}p^{-1}(0)$ | $q^{-1}p^{-1}(1)$ | $q^{-1}p^{-1}(2)$ | $q^{-1}p^{-1}(3)$ | $q^{-1}p^{-1}(4)$ | ...      |
| $q^{-2}p^{-1}(0)$ | $q^{-2}p^{-1}(1)$ | $q^{-2}p^{-1}(2)$ | $q^{-2}p^{-1}(3)$ | $q^{-2}p^{-1}(4)$ | ...      |
| $q^{-3}p^{-1}(0)$ | $q^{-3}p^{-1}(1)$ | $q^{-3}p^{-1}(2)$ | $q^{-3}p^{-1}(3)$ | $q^{-3}p^{-1}(4)$ | ...      |
| $\vdots$          | $\vdots$          | $\vdots$          | $\vdots$          | $\vdots$          | $\ddots$ |

Finally, as  $q^{-1}$  is monotone and  $q^{-1}(x) > p^{-1}(x)$ , we deduce that

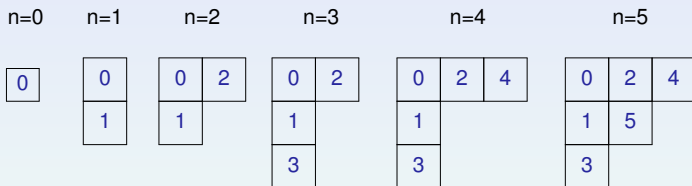
$$\bigcap_{j=0}^{\infty} q^{-j}(\mathbb{N}) = \emptyset$$

and so the embedding of  $P_{\infty}$  is effective.

# From the infinite to the finite

Every DBB Cantor point determines an  $(\infty, \infty, \infty, \dots)$  standard Young tableau. These can be written as sequences of finite standard Young tableaux.

For the alternating Cantor point:



... just a complicated way of counting in binary!



# From Sets to Spaces

Adding in Topology & Category Theory

# The clopen topology

The **Cantor space**  $\mathcal{C}$  is the Cantor set  $\mathcal{C}$  together with the **clopen topology**.

This is generated by the **clopen basis**

$$\{w\mathcal{C} : w \in \{0, 1\}^*\}$$

## Basic open covers

These are determined by some  $R \in \{0, 1\}^*$  where

$$\bigcup_{r \in R} r\mathcal{C} = \mathcal{C}$$

A **minimal** cover is a basic open cover satisfying

$$r\mathcal{C} \cap r'\mathcal{C} = \emptyset \quad \forall r \neq r' \in R$$

# From open covers to prefix codes

Given some  $R \subseteq \{0, 1\}^*$ , then

$R\mathcal{C}$  is a minimal open cover  
iff  
 $R$  is a maximal prefix code.

A relevant fact ...

The set of maximal prefix codes is closed under the induced subset composition.

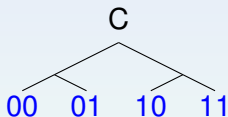
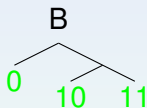
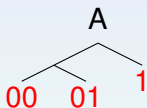
We will concern ourselves with **finite** covers.

# A picture is worth a thousand words ...

There is a well-established bijection between

- (Finite) prefix codes over  $\{0, 1\}^*$
- (Finite) complete binary trees.

$$A = \{00, 01, 1\}, \quad B = \{0, 10, 11\}, \quad C = \{00, 01, 10, 11\}$$



# An uninteresting(?) groupoid

Define the groupoid  $\mathcal{P}$  as follows:

**Objects** All finite maximal prefix codes over  $\{0, 1\}^*$

**Arrows** Bijections of prefix codes that are **monotone** w.r.t. the lexicographic ordering.

This is fairly uninteresting:

There is precisely one arrow between any two prefix codes of the same size.

# What is interesting about $\mathcal{W}$ ?

The groupoid  $\mathcal{P}$  has two distinct categorical tensors.

Given finite, maximal prefix codes  $R, S \subseteq \{0, 1\}^*$ ,

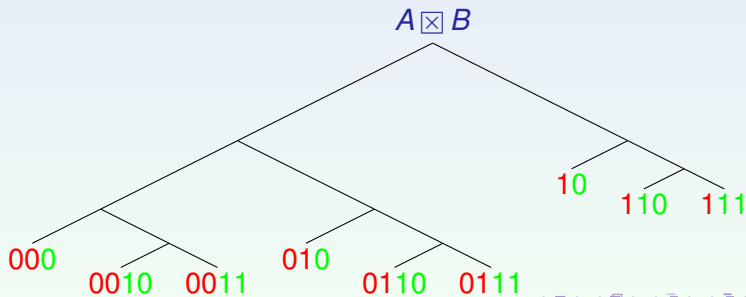
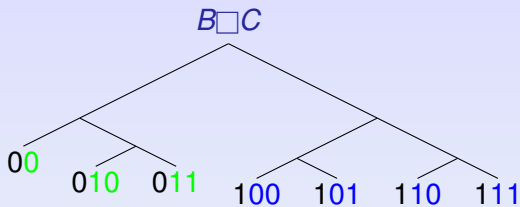
The Multiplicative tensor  $R \boxtimes S = \{rs\}_{r \in R, s \in S}$

The additive tensor  $R \boxplus S = \{0\} \times R \cup \{1\} \times S$

On arrows ...

The tensor on arrows is determined by uniqueness(!)

# As binary trees ...



# A categorical reminder ...

A key structure from the foundations of category theory (MacLane's Theorem):

## MacLane's $(\mathcal{W}, \square)$

**Objects** All finite complete binary trees.

**Arrows** Unique arrow between any two trees of the same rank.

**Tensor** Paste two trees together at their root!

We have an equivalence of categories  $(\mathcal{P}, \square) \cong (\mathcal{W}, \square)$ .

**Question:** What about the 'other tensor' & categorical distributivity?



# Back to Cantor space

Each arrow of MacLane's  $\mathcal{W}$  uniquely determines a homeomorphism of Cantor space:

- Given  $\phi : R \rightarrow S$ , a monotone bijection of finite maximal prefix codes,
- define  $T(\phi) : \mathcal{C} \rightarrow \mathcal{C}$  by:

$$T(\phi)(rw) = \phi(r)w \quad \forall r \in R, w \in \mathcal{C}$$

$T(\phi)$  is:

- Injective, by construction.
- Globally defined, as  $R\mathcal{C}$  is an open cover.
- Surjective, as  $S\mathcal{C}$  is an open cover.
- Continuous — basic open sets map to basic open sets.

# What the F. is this group?

$T()$  is a **faithful functor** from a groupoid to a group.

Its image is closed under *composition* and *inverses*, and contains the identity.

The obvious question:

What is this group of homeomorphisms of  $\mathbb{C}$  ?

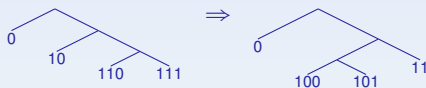
# Some explicit calculations ...

Within the groupoid  $\mathcal{W}$  (equivalently,  $\mathcal{P}$ )

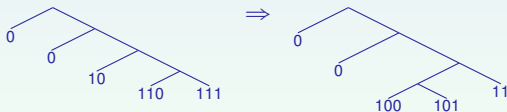
Let  $X_0$  be the unique arrow



Let  $X_1$  be the unique arrow



Let  $X_2$  be the unique arrow



Let  $X_3$  be ...

# Mapping things down to Cantor space ...

Let us then define  $x_j = T(X_j) : \mathcal{C} \rightarrow \mathcal{C}$  for all  $n \in \mathbb{N}$

Simple direct calculation gives:

$$x_i^{-1} x_j x_i = x_{j+1} \quad \forall i < j \in \mathbb{N}$$

We have the *generators* and *relations* of Thompson's group  $\mathcal{F}$

Appealing to the fact that  $F$  has no non-abelian quotients,

The image of  $T(-)$  contains a copy of  $\mathcal{F}$ .

With a little more work ...

The image of  $T(\ )$  is precisely Thompson's  $\mathcal{F}$ .

# More questions than answers

There is a close connection between:

- 1 Minimal basic open covers of Cantor space
- 2 MacLane's  $W$  & the foundations of category theory
- 3 Thompson's group  $\mathcal{F}$ .

*By varying assumptions (finiteness, monotonicity, maximality, &c.)  
we recover many interesting & familiar structures!*

What structures do we recover when we look at  
minimal basic open covers of:

- 1 The Ballot Scott domain?
- 2 Dual-Balanced Ballot Cantor points?