Reflexivity and Self-Referentiality

In Inverse Monoids and Categories

Peter M. Hines

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Some motivation ...

This talk is about some **inverse category theory**

closely associated with **logic** and **theoretical computer science**.

The general topic is **models of self-referentiality**.

We aim to:

1. Describe the historical context & importance.
2. Give concrete axioms & examples.
3. Do all this in the reversible (inverse monoid) setting.

Historical Context (I) — Foundations & Logic

Scenes from the frog-mouse wars

The historical setting

The late 19th and early to mid 20th century saw something of a crisis in the foundations of mathematics.

This can be compared to the controversy caused by the introduction of calculus that was resolved by rigorous notions of limit & convergence.

However, it was more profound, and less easily resolved. Its aftermath is still relevant today.

The problems of infinity

Georg Cantor lit the fuse, and stepped back to a safe distance ...

His work was not always appreciated:

> *A "scientific charlatan", a "renegade" and a "corrupter of youth"* — **Leopold Kroenecker**

> *Mathematics is "ridden through and through with the pernicious idioms of set theory", which is "utter nonsense" that is "laughable" and "wrong"* — **Ludwig Wittgenstein**

A more balanced approach

A very readable contemporaneous account:

Mathematical Rigor, past and present
– J. Pierpont (1928)

The Mengenlehre of Cantor *[Set Theory]* has brought to light a number of paradoxes which have profoundly disturbed the mathematical community for a quarter of a century. Mathematical reasoning which seemed quite sound has led to distressing contradictions. . . . **there is no guarantee that other forms of reasoning now in good standing may not lead to other contradictions as yet unsuspected.**

A quick reminder ...

Cantor's diagonal argument:

$$\begin{array}{ll}
s_1 & = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
s_2 & = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \ldots \\
s_3 & = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \ldots \\
s_4 & = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_5 & = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ \ldots \\
s_{11} & = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}$$

$$s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ \ldots$$

The natural numbers are countable – the unit interval is not.

The first problem: Non-constructive existence proofs

A very simple result (G. Cantor, unpublished)

The real line is uncountable; the set of algebraic numbers is countable. Therefore, ‘nearly all’ numbers are transcendental.

This was *not* (despite common claims) part of Cantor’s 1874 paper.

> *“The restriction which I have imposed on the published version of my investigations is caused in part by local circumstances”* – G. Cantor, letter to R. Dedekind

> *“Kronecker uses his authority to support the view that all who have labored on foundations . . . are sinners before the Lord”* – K. Weierstrauss

Paradoxes in naive set theory

Objections were not (just) based on personal bias:

Cantor (and others) used what we now call naive set theory.

The axiom of comprehension

Given a predicate $\Phi(x)$, we may form the set containing precisely those objects for which Φ is true: $\{x : \Phi(x)\}$.

Naive set theory is inconsistent:

Russell's paradox (1901)

Let us define the set of all sets that are not members of themselves ...

Historical curiosity: Cantor himself was known to be skeptical about this axiom.

The origins of Russell's paradox

1. Russell claimed to have discovered this by analysing *diagonalisation arguments*.
2. It was also strongly based on Cesare Burali-Forti's ordinals paradox (1897):

   > *"The set of ordinals is well-ordered, so must have an ordinal. This ordinal must be both an element of the set of all ordinals and yet greater than every such element. "*

Cantor was also aware of the cardinality paradox:

> "what is the cardinality of the set of all cardinals?"

Cantor's work was based on *inconsistent foundations* ...

unfortunately, so was much of the rest of mathematics.

One unfortunate consequence

"The Foundations of Arithmetic: a Logico-Mathematical Enquiry"

— Gottlob Frege (1903)

Part (II) in press when Frege received a letter from B. Russell.

Russell, on Frege: (!?)

"His entire life's work was on the verge of completion . . . upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. "

Several modest proposals

The difficulty was to find a system of foundations that:

- Ruled out any of the growing list of paradoxes.
- Did not also rule out much of established mathematics!
- Did / did not allow for Cantor's constructions (according to personal bias ..)

This program did not proceed smoothly.

A modest proposal

Hilbert's Formalism Program:

- An axiomatic formalisation of all mathematics
- A proof of consistency by purely 'finitary' methods
- The "Entshceidungsproblem" – a systematic method of deciding the truth of any proposition from these axioms.

Mathematics as a formal game? – a common criticism

Propositions are decided by formal syntactic manipulations — there is no rôle for intuition about sets / infinity / real or natural numbers, &c.

Hilbert's program eventually collided with K. Gödel's incompleteness theorems.

A problem with logic?

The polar opposite was the Intuitionism of Brouwer, Heyting, Kolmorogov, &c.

The untrustworthiness of the principles of logic – Brouwer (1908)

- The philosophical position: Mathematics is about mental constructions, not external truth.
- An immediate consequence: Non-constructive existence proofs are not valid.
- The concrete implementation: The laws of logic need to be changed, to reject:
  - The law of the excluded middle:

    $$A \vee \neg A \;\Leftrightarrow\; \mathit{True}$$

  - Double negation

    $$\neg(\neg A) \;\Leftrightarrow\; A$$

  - (Implicitly) Proof by Contradiction, and the Axiom of Choice.

The Batrachomyomachia [frog-mouse wars]

Hilbert & Brouwer clashed, in a fierce manner.

Intuitionism would rule out, not only Cantor's work, but Hilbert's Basis Theorem (1890)

– a non-constructive existence proof, using L.E.M.

"To prohibit existence statements and the principle of excluded middle is tantamount to relinquishing the mathematics altogether."
– D. Hilbert 1927

(1928-29) Dissolution & re-forming of editorial board of Annals of Mathematics.

Axiomatic set theory

A modern consensus(?): Z.F.C.

- Zermelo-Fraenkel set theory, with the Axiom of Choice.
- Developed from 1908 onwards . . .
- ‘Generally accepted’ as a foundational system by mid 1960s.

Cannot (c.f. Gödel’s theorems) prove its own consistency.

Sets and Classes

Z.F.C. is clear about what it does, and does not, describe.

We need to step outside ZFC to discuss, for example, properties of *all monoids*, or *all inverse semigroups*, &c.

This may be done using the von Neumann – Gödel – Bernay theory of proper classes.

— a significant precursor to Category theory.

Cantor's 'set of all cardinals' is in fact a proper class.

The problem of impredicativity

Henri Poincaré was considered a ‘less radical’ intuitionist, who viewed paradoxes as caused by the use of impredicative definitions”

Impredicative definition:

A definition that references the object being defined ...

Poincaré & Russell’s vicious circle principle:

No object or property may be introduced by a definition that depends on that object or property itself.

Neatly ruling out:

“The set of all sets that do not contain themselves”.

This seems more reasonable than re-thinking the whole of logic!

Impredicativity lurks in unexpected places:

- The greatest lower bound of a set of numbers.
- Dedikind cuts (construction of real numbers from the rationals).

As pointed out by Zermelo:

This happens, for example, in the well-known Cauchy proof of the fundamental theorem of algebra, and up to now it has not occurred to anyone to regard this as something illogical".

Embracing Impredicativity

A strong defence of impredicative definitions was given in:

"The logicist foundation of mathematics" – R. Carnap (1931)

A hard problem:

When can we take an impredicative definition, and give a predicative one?

- The function $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(n) = n.f(n-1)$
- The object in a closed category isomorphic to its own function space.

A strong (& controversial) claim:

"There is no algorithm for building something that is impredicative"
– Essays on Life Itself (R. Rosen, 1999)

Impredicativity in other fields (I)

Computer Scientists are quite happy to base concrete programming languages on impredicativity:

RECURSION

- A Lisp version of the Factorial function

```
(defun fact (n)
   (cond ((zerop n) 1)
         (t (* n (fact (- n 1))))))
```

- For example:

Factorial(5) = 5 X Factorial(4)
= 5 X 4 X Factorial(3)
= 5 X 4 X 3 X Factorial(2)
= 5 X 4 X 3 X 2 X Factorial(1)
= 5 X 4 X 3 X 2 X 1 X Factorial(0)
= 5 X 4 X 3 X 2 X 1 X 1

```
Break 2 [13]> (fact 6)
1. Trace: (FACT '6)
2. Trace: (FACT '5)
3. Trace: (FACT '4)
4. Trace: (FACT '3)
5. Trace: (FACT '2)
6. Trace: (FACT '1)
7. Trace: (FACT '0)
7. Trace: FACT ==> 1
6. Trace: FACT ==> 1
5. Trace: FACT ==> 2
4. Trace: FACT ==> 6
3. Trace: FACT ==> 24
2. Trace: FACT ==> 120
1. Trace: FACT ==> 720
720
```

4

Impredicativity in other fields (II)

Linguists are well aware that dictionaries are necessarily impredicative:

Webster's Dictionary

- Hill — "a usually rounded natural elevation of land lower than a mountain"
- Mountain — "a landmass that projects conspicuously above its surroundings and is higher than a hill"

Impredicativity in other fields (III)

In Natural Language Processing, the ‘meaning’ of a word is simply its statistical relationship with other words in some set of documents:

“Word vectors are simply vectors of numbers that represent the meaning of a word.” – Introduction to Word Vectors (J. Ahire)

Frequently justified by reference to:

- “You shall know a word by the company it keeps” – J. Firth
- “The meaning of words lies in their use” – L. Wittgenstein

In A.I. ...

Searle’s ‘Chinese Room’ thought experiment is based on rejecting the claim that understanding meaning is simply knowing the relationship between words, with no external point of reference

Back to foundations ...

"There is just one point where i have encountered a difficulty. **You state that a function can also act as an indeterminate element** [a variable]*. This I formerly believed, but now seems doubtful to me because of the following contradiction ..."*

— Russell's 1902 letter to Frege:

A (very) radical solution(!)

Alonzo Church (1932-33) proposed a single system for both *logic* and *foundations* that **erased the distinction between functions and their arguments**.

(This would involve embracing impredicativity in a major way).

Wrong, but for interesting reasons ...

The purely logical part of Church's system was shown to be inconsistent:

The Kleene-Rosser paradox (1935)

- Based on Richard's paradox (a form of Cantor diagonalisation)
- Cited by Gödel as essentially equivalent to his first incompleteness theorem.

The core dealing with foundations was a consistent — the (untyped) lambda calculus.

Historical Context (II)

— Computability & Decidability

Where the abstract becomes concrete

The lambda calculus

Defining λ-terms

Start with an infinite set V of **variables**.
Proceed inductively

- **Variables** Any $v \in V$ is a lambda term.
- **Application** Given lambda terms M, N, then (MN) is a λ term.
- **Abstraction** Given a variable v and a lambda term M, then $\lambda v.M$ is a λ term.

Questions of notation

Conventions:

- Assume left-bracketing for application

$$M_1 M_2 M_3 M_4 \text{ is shorthand for } (((M_1 M_2) M_3) M_4)$$

- Assume right-bracketing for abstraction:

$$\lambda x.MN \text{ is shorthand for } \lambda x.(MN)$$

$$\lambda x_1 x_2 x_3.M \text{ is shorthand for } \lambda x_1.(\lambda x_2.(\lambda x_3 M))$$

How should we interpret λ terms – what can we do with them??

Bound and Free variables

In a lambda term, variables are either bound or free,
depending on whether they fall under the scope of some $\lambda\_.(\ldots)$.

$$(\lambda gf.(gx)(fy))(\lambda ba.h(ba))$$

The variables a, b, f, g are bound by the λ operations.
The variables h, x, y are free variables.

Combinators ...

A term with no free variables is a **combinator**.

Equivalences of lambda terms

The names of bound variables are **unimportant**.

α-equivalence

We may re-name bound variables,
provided there is no clash with free variable names.

$$\lambda xy.x(yx) \quad \equiv_\alpha \quad \lambda ab.a(ba)$$

We use alpha-equivalence to avoid ‘clashes’.

Dynamics of lambda terms

The dynamics are based on substitution:

Given a lambda term M, let us replace every occurrence of x by N ...denote the result by $M[N/x]$.

The key operation: the β**-step**.

$$(\lambda x.M)(N) \ \rightarrow_\beta \ M[N/x]$$

$$(\lambda xy.xyx)(\lambda pq.ppq) \ \rightarrow_\beta \ \lambda y.(\lambda pq.ppq)y(\lambda pq.ppq)$$

Caution:

We repeatedly rely on α-equivalence to avoid clashes of variable names. The syntactic rules for this are subtle!

Reduction, confluence, and equivalence

The key notion is β-reduction:

$$M \rightsquigarrow_\beta N$$

when N can be derived from M via a (finite) series of β-steps.

Confluence: the Church-Rosser Property

Given distinct N_1 and N_2, where

$$M \rightsquigarrow_\beta N_1 \text{ and } M \rightsquigarrow_\beta N_2$$

There exists some N such that

$$N_1 \rightsquigarrow N \text{ and } N_2 \rightsquigarrow N$$

Terms M, N are β-**equivalent**, written $M =_\beta N$ if there exists some P with

$$M \rightsquigarrow_\beta P \text{ and } N \rightsquigarrow_\beta P$$

Encoding natural numbers

The natural numbers are straightforward:

Church numerals:

- $\overline{0} = \lambda fx.x$
- $\overline{1} = \lambda fx.fx$
- $\overline{2} = \lambda fx.f(fx)$
- $\overline{3} = \lambda fx.f(f(fx))$
- $\ldots$

Encoding arithmetic

As are arithmetic operations:

successor function $\lambda nfx.f(nfx)$

addition $\lambda abfx.(af)(bfx)$

multiplication $\lambda abfx.a(bf)x$

exponentiation $\lambda abfx.abfx$

predecessor Left as an exercise ...

Question:

Which operations can, and cannot,
be described using λ-terms?

Fixed points & combinators

A useful – albeit bizarre – feature:

We can find a **fixed point** for *any* λ-term.

Turing's fixed point combinator

Let us define $\Omega = \lambda xy.y(xxy)$, and $\mathbf{Y} = \Omega\Omega$.

Observe that $\mathbf{Y}f$ is a fixed point for f:

$$f(\mathbf{Y}f) \ =_\beta \ \mathbf{Y}f$$

How useful / odd are fixed points?

A positive feature:

We can use the fixed point combinator to write down (e.g.) the factorial function, simply from its fixed-point definition.

A bizarre feature

Everything has a fixed point, including:

- The successor function.
- Church numerals.
- The fixed point combinator ...

  Exercise: What is this?

Back to foundations, for a while

Part of Hilbert's program was the Entscheidungsproblem:

"An **effective procedure** to decide the truth or falsehood of a proposition within an axiomatic formalisation of mathematics".

An undefined term

What is an 'effective procedure'?

Effective procedures ...

Which functions $f : \mathbb{N} \to \mathbb{N}$ on the natural numbers

can be described using an effective procedure?

Can we give an algorithm for any function $f : \mathbb{N} \to \mathbb{N}$?

A simple cardinality argument would suggest otherwise!

The impact of dentistry on mathematics

When Church & his student Kleene were exploring how to encode arithmetic in λ calculus, the *predecessor* function was particularly problematic.

$$(Pred\ \overline{n}) =_\beta \begin{cases} \overline{n-1} & n > 0 \\ \overline{0} & \text{otherwise.} \end{cases}$$

> *Kleene described how he found the solution while being anesthetized by laughing gas (nitrous oxide) for the removal of four wisdom teeth. After Kleene showed the solution to his teacher, Church remarked something like:* **"But then all intuitively computable functions must be lambda definable. In fact, lambda definability must coincide with computability"**
>
> *The impact of λ calculus in logic & C.S.*
> *— H. Barendregt (1997)*

Other notions of computability

- λ calculus, characterisation of computability, in: "An unsolvable problem of elementary number theory" (1936)
- The Turing machine characterisation of computability, in: "On computable numbers, with an application to the Entscheidungsproblem" (1936)

Shown to capture the same class of functions by A. Turing:

- Computability and λ-definability (1937)

The Church-Turing thesis

The notion of λ-definability (equivalently - T.M. computable) captures the intuitive notion of "effective procedure".

Historical note: K. Gödel was skeptical about λ calculus capturing the notion of "effective procedure", until Turing's 1937 paper.

Models of λ calculus??

An important question:

Application looks like / is designed to model
"applying a function to an argument".

Can this be made precise?

1. λ calculus is a formal, syntactic system. Can it have any concrete (semantic) models within ZFC?
2. If so, shouldn't Turing machines have similar / identical models??

Models of the λ calculus

A (previously) common viewpoint:

It is hard to believe in a sets-and-functions model of λ calculus.

- In λ calculus, functions and arguments are the same thing ...

  The set of functions on a set is not the same cardinality as the set itself!

  $\{f : \mathbb{N} \to \mathbb{N}\} \ncong \mathbb{N}$ **(This is where all the trouble started!)**

- Can we really have fixed points for *arbitrary* functions?

  What about all those paradoxes?

- Can we relate predicative and impredicative definitions?

Until late 1960s / early 1970s, λ calculus was simply a formal, syntactic system.

The sticking point ... and a solution

For any set $\mathcal{D}$, the set of functions $Set(\mathcal{D}, \mathcal{D})$ is not the same cardinality as $\mathcal{D}$ itself.

Dana Scott's solution

- Let $\mathcal{D}$ be a **topological space**.
- Consider the *continuous* functions from $\mathcal{D}$ to itself, $Top(\mathcal{D}, \mathcal{D})$.
  - This may have *lower cardinality* than the set of all functions on $\mathcal{D}$.
  - It may, provided $\mathcal{D}$ is 'well-behaved', also be a topological space.
- It seems *possible* to have the same type for *operations* and their *arguments*.

It is possible to work with order theory only, and 'hide' the topology.

Some rather technical order theory

Scott domains – rather subtle order-theoretic structures!
Partially ordered sets $(\mathcal{D}, \leqslant)$ where:

- **$\mathcal{D}$ has a bottom element $\perp \in \mathcal{D}$.**
- **Every directed set has an upper bound.**

  (A directed subset $A \subseteq \mathcal{D}$ is one where every pair of elements $a, b \in A$ has an upper bound $a, b \leqslant c \in A$).
- **If a subset $S \subset \mathcal{D}$ has an upper bound, it has a *least upper bound*.**
- **Every element $d \in D$ is the supremum of a directed subset of "finite" elements.**

  An element $f \in \mathcal{D}$ is "finite" when, for every directed subset $A \subseteq \mathcal{D}$

$$f \leqslant Sup(A) \ \Rightarrow \ \exists a \in A \ s.t. \ f \leqslant A$$

Models of the λ calculus

A function $f : \mathcal{D} \to \mathcal{D}$ is **Scott-continuous** when:

- $f(\bot) = \bot$
- $a \leqslant b \;\Rightarrow\; f(a) \leqslant f(b)$
- $f\,(Sup(A)) = Sup\,\{f(a) : a \in A\}$, for all directed sets $A \subseteq \mathcal{D}$.

As a consequence:

- **Kleene's fixed-point theorem**
  Every S.-C. function has a **least fixed point**.
- The set $[\mathcal{D} \to \mathcal{D}]$ of S.-C. functions on $\mathcal{D}$ is itself a Scott domain.
- $[\mathcal{D} \to \mathcal{D}]$ is a sub-domain of $\mathcal{D}$.

A recent (2016?) result

'Almost all' Scott domains provide lambda models.

λ-models from Cantor space

An example, from: “Notes on the category theory of Cantor space” (PMH – in progress):

Cantor space $\mathfrak{C}$ is the set of one-sided infinite strings over $\{0, 1\}$
A point of Cantor space $c \in \mathcal{C}$ is **ballot** when every prefix u satisfies:

$$\#1s \text{ in } u \ \leqslant \ \#0s \text{ in } u$$

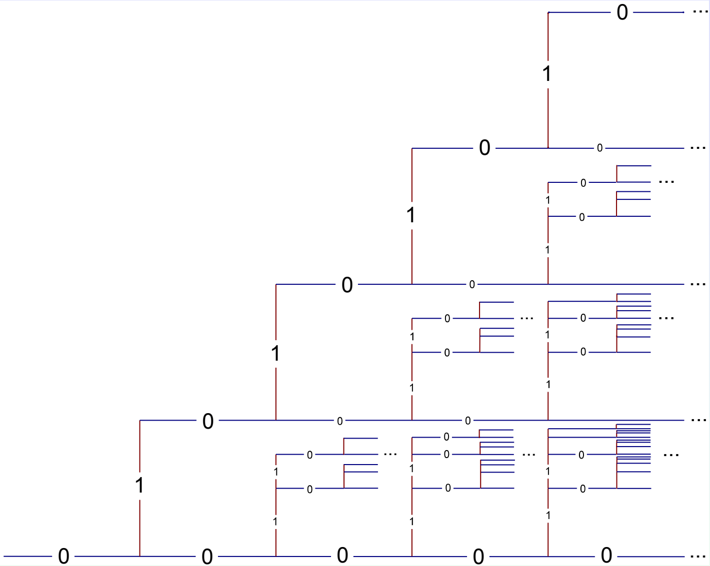
Take the pointwise partial order on Ballot Cantor points:

$$a_0 a_1 a_2 a_3 \ldots \ \leqslant \ b_0 b_1 b_2 b_3 \ldots \ \text{ iff } \ a_j \leqslant b_j \ \forall j \in \mathbb{N}$$

This particular Scott domain is a subset of Cantor space.

We can draw a picture.

The ballot Cantor Scott domain



The ballot Cantor Scott domain

Other, equivalent, formalisations.

Points of this space (excluding a subset of measure 0) also correspond to:

- (∞, ∞) standard Young Tableaux
- Effective representations of Nivat & Perot's polycyclic monoid P_2 as monotone partial injections on $\mathbb{N}$.
- A class of categorical tensors.

This is the subject of a different talk(!)

(PMH — NBSAN — York, Jan. 2019)

A Modern Updating (I)

— Category Theory & Logic

Finding closure with proofs and programs

An unexpected connection ...

λ **calculus** has a surprising deep connection to **intuitionistic logic**.

The Curry-Howard correspondence

A correspondence between *logical* and *computational systems*.

Also known as the “Proofs-as-Programs” correspondence.

H. Curry (1958) Hilbert deductions systems / typed combinatory logic

W. Howard (1969) Intuitionistic logic / typed λ calculus.

Curry-Howard has been extended beyond intuitionistic logic, although computational interpretations of ‘classical logic’ (natural deduction) are somewhat unsatisfactory.

How far it may be extended is an open question

A (very) brief overview of types

Types in λ calculus restrict allowable applications.
In 'simply typed' systems, types are freely built by induction from some basic set.

- If M has type s and x is a variable of type t, then

$$\lambda x.M \quad \text{has type} \quad s \rightarrow t$$

- If P has type $s \rightarrow t$ and Q has type s, then

$$PQ \quad \text{has type} \quad t$$

Untyped lambda calculus is where everything is the same type!
Simply typed, and untyped, are polar opposites.

Proofs as Programs as Closed Categories

The Curry-Howard-Lambek-Scott correspondence:
The ‘rules’ for intuitionistic logic / typed lambda calculus are ‘the same’ as those for Cartesian closed categories ...

Propositions in Intuitionistic Logic

$\equiv$

Types in λ calculus

$\equiv$

Objects in a Cartesian closed category

Cartesian Closed Categories:

The basic idea: the 'function space' of arrows between objects is also an object in the same category.

The axiomatisation

For any objects A, B in $Ob(\mathcal{C})$, we have:

- The Categorical Product $A \times B \in Ob(\mathcal{C})$
- The **Internal Hom.** $[A \rightarrow B] \in Ob(\mathcal{C})$.

The *internal hom* acts like $\mathcal{C}(A, B)$
i.e. the collection of arrows from A to B.

There exists an **evaluation map**

$$eval_{A,B} \in \mathcal{C}(A \times [A \rightarrow B], B)$$

Satisfying several natural, but subtle, axioms.

Eliminating types

"Introduction to Higher-Order Categorical Logic"
– J. Lambek & P. Scott (1986)

Models of the (pure, untyped) lambda calculus given by C-monoids:

Cartesian Closed Categories with only one object.

$$\mathcal{D} \cong \mathcal{D} \times \mathcal{D} \cong [\mathcal{D} \to \mathcal{D}]$$

- Explicit axioms given in L.-S.
- Concrete examples ... not easy to construct.
  Scott domains are a useful source!

What is needed

A C-monoid is the endomorphism monoid of some object D in a CCC. This object must satisfy:

Self-similarity $D \cong \mathcal{D} \times \mathcal{D}$

This is easy to satisfy:

- Hilbert's 'Grand Hotel' parable.
- Fractals such as Cantor space.
- Any countably infinite set (in a suitable category).

Reflexivity $\mathcal{D} \cong [D \to \mathcal{D}]$

This is much harder to satisfy

- Simply does not happen for sets and functions
- We need hom-sets restricted by (for example) topological, or order-theoretic, properties.

Possibly relevant

A useful tool:

> *"Coherence and Strictification for Self-Similarity"*
> *– Journal of Homotopy & Related Structures (PMH 2017)*

A coherence theorem giving an **equivalence of categories** between multi-object categories and the single-object (i.e. monoid) case.

Equivalences of categories preserve all relevant categorical structure.

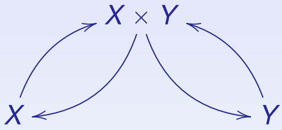
A great deal of interesting algebra arises as monoid versions of fundamental categorical properties.

Mapping category theory to algebra

The multi-object case: Coherence for associativity

The single-object case: Richard Thompson's group $\mathcal{F}$

The multi-object case: Projections & Injections for a product



The single-object case: Nivat & Perot's polycyclic monoid P_2.

More details in:

- Categorical coherence in cryptography (PMH, under review)
- Notes on the Category theory of Cantor space (In progress ...)
- York security group talk (PMH, Feb. 2019)
- The categorical theory of self-similarity (PMH 1999)

A Modern Updating (II)

— Other Logics & Categories

Cartesian is not the only closure

Linear Logic

Linear Logic: introduced by J.-Y. Girard in 1987.

Stanford Encyclopedia of Philosophy:

- A refinement of both classical and intuitionistic logic.
- Combines the symmetry of classical logic with the construtivism of intuitionistic logic.
- Emphasises resource-sensitivity: structural rules are explicit, and restricted!
  - As a consequence: operations are **reversible**.

Gives a refinement of logical operations:
Two subtly different forms of conjunction and disjunction.
Explicit operations for propositions that can be copied / deleted.

An interesting model of linear logic

The Geometry of Interaction (I) — J.-Y. Girard (1989)

- A 'rather original' model of (a fragment of) Linear Logic.
- The start of a long series of GoI papers ...
- Based on C^* algebras over Cantor space, with deduction interpreted by analytic formulæ
- Closely related to a study of λ-calculus:

  "Local and asynchronous β-reduction" — V. Danos & L. Regnier (1992)

  – in particular, the same underlying algebra.

Some ‘peculiarities’ of GoI

Somewhat degenerate as a logical model ...

- “This operation is used to model both conjunction and disjunction”
- “Propositions are identified with their negations.”
- “Our system forgets types”

Strongly suggesting a computational, rather than logical, interpretation.

Interpreting Girard

As noted by many people:

- The "C^*-algebras over Cantor space" were not necessary;

  the same system could be built using **partial injections** on countable sets.

  — the algebraic basis was therefore inverse semigroups: precisely, Nivat & Perot's polycyclic monoid P_2, known to logicians as the **dynamical algebra**.

- The closed categories used were **compact closed**, rather than **Cartesian closed**.

S. Abramsky 1996, PMH 1997, M. Hyland (unpublished), E. Haghverdi & P. Scott 2000, &c.

Compact closed categories

The same basic idea: the 'function space' of arrows between objects is also an object in the same category.

1. We have a symmetric tensor $\_ \otimes \_ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.
   - We do not have the diagonals / projections associated with a Cartesian product.
   - There is a unit object satisfying $I \otimes X \ \cong \ X \ \cong \ X \otimes I$
2. There is a contravariant **dual** $(\ )^* : \mathcal{C} \to \mathcal{C}$ satisfying $((\ )^*)^* = Id_{\mathcal{C}}$.
3. The **internal hom** is defined in terms of the dual:

$$[A \to B] \ \cong \ A^* \otimes B$$

4. The **evaluation maps** are defined in terms of simpler operations:

$$\eta_A : I \to A \otimes A^* \quad \text{and} \quad \epsilon_X : A^* \otimes A \to I$$

that satisfy: $(1_A \otimes \eta_{A*})(\epsilon_{A*} \otimes 1_A) = 1_A$.

A paper by a student ...

"A compact closed inverse monoid used in the Geometry of Interaction"
(PMH 1996) – unpublished, but referenced.

Relevant parts later appeared in:

- "The algebra of self-similarity and its applications" (PMH 1998)
- "The categorical theory of self-similarity" (PMH 1999)
- "A categorical framework for finite state machines" (PMH 2003)

The key observation

The **internal hom** is defined in terms of the dual:

$$[A \to B] \;\cong\; A^* \otimes B$$

When we have **self-similarity** *(The easy property)*
we get **reflexivity** *(The difficult part)*
for **free.** *(That's useful!)*

Let $N \cong N^*$ be a self-dual object satisfying $N \cong N \otimes N$.

$$[N \to N] \;\cong\; N^* \otimes N \;\cong\; N \otimes N \;\cong\; N$$

Self-duality is not an issue: $A \otimes A^*$ is isomorphic to its dual, for any object A.

A concrete construction

A concrete construction (the **Int** construction) of compact closed categories was given, as *pure category theory* in

Traced monoidal categories
A. Joyal, R. Street, D. Verity (1996)

An equivalent construction (the **GoI** construction), based on C.S. and logic was given in

Retracing Paths in the Process Algebra
S. Abramsky (1996)

A generic construction, giving **all** compact closed categories.

A concrete example, from J.S.V.

The category **IntRel**, based on the category **Rel** of relations ...

Objects These are pairs of sets (X, U)

Arrows An arrow from (X, U) to (Y, V) is a relation from $X \uplus V$ to $Y \uplus U$.
Usually written as a matrix

$$\begin{pmatrix} r_{00} & r_{01} \\ r_{10} & r_{11} \end{pmatrix} : X \uplus V \rightarrow Y \uplus U$$

Composition This is kind of complicated ...

Tensor Defined by:

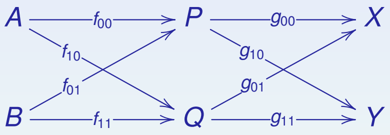
$$(X, U) \otimes (A, P) \;=\; (X \uplus A, P \uplus U)$$

Composing matrices of relations

Given relations

$$\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} : A \uplus B \to P \uplus Q \ \text{ and } \ \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} : P \uplus Q \to X \uplus Y$$

Composition is given by the usual 'summing over paths'



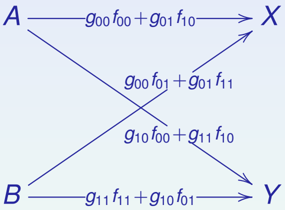
Note: matching indices in the subscripts ... $g_{01} f_{10}$ **,** $g_{00} f_{01}$ **&c.**

Composing matrices of relations

In the category **Rel** of relations

$$\begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} : A \uplus B \to P \uplus Q \text{ and } \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} : P \uplus Q \to X \uplus Y$$

Composition is given by the usual 'summing over paths'



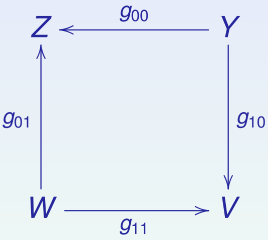
Is composition uniquely determined by matching subscripts? $g_{01} f_{10} + g_{00} f_{00}$ **, &c.**

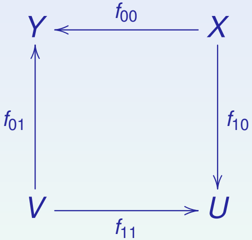
Another ‘summing over paths’ construction

In the category **IntRel**, we draw arrows

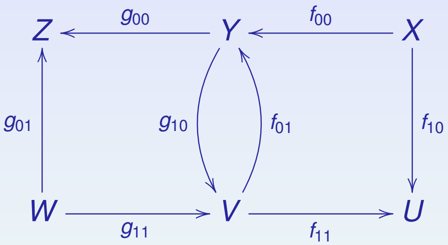
$$G : (V, Y) \to (Z, W) \text{ and } F : (X, U) \to (Y, V)$$

as **planar** 4-tuples of relations:





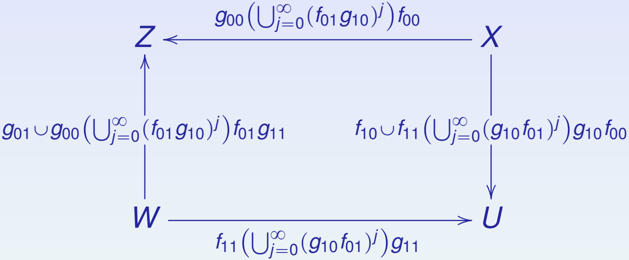
Let us bring these together, in the only way possible



and again ‘sum over paths’.

Composition in **IntRel**

The intuition is simple ...



even though the formulæ appear complex!

Some notable points

- This gives exactly (up to some C^*-algebra window-dressing) the formula used – without explanation – by Girard to model deduction.
- All this may be done with **partial injections** rather than relations.
- The object $(\mathbb{N}, \mathbb{N})$ is self-similar.
- Every object of the form (X, X) is self-dual.

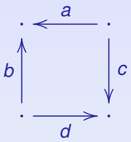
The monoid of arrows on $(\mathbb{N}, \mathbb{N})$ is indeed a “compact closed monoid”, as used by J.-Y. Girard.

What, exactly, is being modeled??

- **J.-Y. Girard (1989)**
  System $\mathcal{F}$ – a **'polymorphically' typed** λ **calculus** where the operations of application and abstraction are applicable to types.
- **S. Abramsky, E. Haghverdi, P. Scott (2004)**
  A form of **untyped combinatory logic** — a lower-level system upon which untyped λ calculus may be built.
- **PMH 2003 / 2008**
  The same algebra describes **Turing machine** computations.

The Turing machine connection

Consider an arrow in **IntRel**



built up from a Turing machine operating on a **short section of** tape.

The interpretation

The elements a, b, c, d are relations / partial injections describing:

- a – all computations that start on the rhs and finish on the lhs.
- b – all computations that start and finish on the lhs.
- c – all computations that start qnd finish on the rhs.
- d – all computations that start on the lhs and finish on the rhs.

What about that composition?

Consider two such 4-tuples, for different sections of tape.
Their composite in **IntRel** is the 4-tuple describing
the behaviour of the TM on the composed, longer section of
tape.

Different routes to the same place:

- Proved in (PMH 2003) by brute-force calculation.
- Proved (& generalised) in (PMH 2008), using an interpretation of TMs via Scott domains.

Used in PMH 2010 to solve an outstanding problem in QM computing

– the subject of a different talk!

Some work in progress

Compact closed monoids (CCMs) are endomorphism monoids of reflexive objects . . . does this mean they are isomorphic to their own endomorphism monoids??

This cannot be the case!

CCMs within **Int(pInj)** have generalised inverses.
— there is no generalised inverse of the homomorphism

$$f(m) = 1 \quad \forall m \in M$$

A category mistake:
The endomorphism monoid of a reflexive object
need not be a reflexive object
in the category of monoids.

A potentially interesting setting ...

Let us restrict endomorphisms ... this time by (partial) reversibility.

A potential definition

Define the category **pinMon** of 'partial inverse monoid monics':

Objects These are all inverse monoids

Arrows An arrow $f : M \to N$ is a partial injection satisfying:

1. $f(a)$ and $f(b)$ are defined implies $f(ab)$ is defined, and equals $f(a)f(b)$.
2. $f(1_M)$ is defined, and equal to 1_N.
3. $f(a)$ is defined implies $f(a^{-1})$ is defined and equals $f(a)^{-1}$.

Reflexivity all the way up??

Endomorphism monoids in **pinMon** are inverse monoids

— although arbitrary homsets are not.

Our aim: "higher-order" reflexivity

Can endomorphism monoids of reflexive objects themselves be reflexive objects, in the right setting?